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Subcouples of codimension one and interpolation of operators that almost agree

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Abstract

Suppose that $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ are Banach couples and suppose that $T_0 : X_0 \to Y_0$ and $T_1 : X_1 \to Y_1$ are bounded and linear. Also assume that $\Gamma \in (\Delta(\bar{X}))'$ and that T_0 and T_1 agree as maps from $\Delta(\bar{X}) \cap \ker \Gamma$ to $\Sigma(\bar{Y})$. If the maps do not agree as maps from all of $\Delta(\bar{X})$ we cannot interpolate T_0 and T_1 to a map $T : J_{\theta,p}(\bar{X}) \to J_{\theta,p}(\bar{Y})$, where $J_{\theta,p}$ denotes the classical J-method. This situation can for example be found in an article on interpolation of Hardy-type inequalities by Krugljak, Maligranda and Persson. We will in this paper define functors $J_{\theta,p;\Gamma}$ such that T_0 and T_1 interpolate to a map $T : J_{\theta,p;\Gamma}(\bar{X}) \to J_{\theta,p}(\bar{Y})$. The main purpose of this paper is to make the definition of the $J_{\theta,p;\Gamma}(\bar{X})$ spaces and build a theory for them. We will also do this for more general real parameters. If Γ is bounded on X_0 it holds that $J_{\theta,p;\Gamma}(\bar{X}) = J_{\theta,p}(X_0 \cap \ker \Gamma, X_1)$. These spaces have been studied by Kalton, Ivanov and Löfström. Their results will follow as corollaries to the more general results of this article and our new theory can be thought of as a theory for generalized subcouples of codimension one.

In the last section, we apply our theory to a situation considered by Krugljak, Maligranda and Persson in connection with Hardy-type inequalities. We prove new results and provide a new way of understanding that kind of problems.

MSC: 46M35

Keywords: Interpolation; Banach space; Banach couple; Subspace; Subcouple; Quotient couple

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1. Introduction

In Section 1, we will give the reader an introduction to the content of the article. For an introduction to interpolation theory see Section 2 and for further reading on that topic see [1–3,6].

If $\bar{X}$ and $\bar{Y}$ are Banach couples such that $Y_0 \subset X_0$ and $Y_1 \subset X_1$ are closed and Y_0 and Y_1 are embedded into $\Sigma(\bar{X})$ with the embeddings into X_0 and X_1 then we say that $\bar{Y}$ is a subcouple of $\bar{X}$. Pisier [15] studied a case where $X_0 = L^p$, $X_1 = L^q$ on the unit circle and $Y_0 = H^p$, $Y_1 = H^q$ on the unit disk. He proved that $\bar{Y}$ is K-closed, that is, it holds that

$$K(t, y, \bar{Y}) \leqslant CK(t, y, \bar{X}) \quad \forall y \in \Sigma(\bar{Y}).$$

From that he concluded that

$$\bar{Y}_{\theta,p} \approx \bar{X}_{\theta,p} \cap \Sigma(\bar{Y})$$

for all θ and p. To prove that $\bar{Y}$ is K-closed he used duality results between subcouples and quotient couples and that duality was later investigated in a more general situation by Janson [5]. To the best of the author's knowledge, interpolation of subcouples and interpolation of quotient couples was first considered by Lions and Magenes [10] and the K-closed concept was first used by Peetre [14].

Löfström [13] looked at different ways of constructing subcouples and one of them was to consider a finite set $\Gamma \subset X_1'$ and let $Y_0 = X_0$ and $Y_1 = X_1 \cap_{\gamma\in\Gamma} \ker\gamma$. In [11] he used those results to interpolate boundary-value problems of Neumann type and in [12] he applied them to interpolation with constraints. He also considered the special case when Γ is just one linear functional on X_1. This case was also independently considered by Ivanov and Kalton and in [4] they published a complete answer, for regular couples, to the question about when $(X_0, X_1 \cap \ker\Gamma)_{\theta,p}$ is a closed subspace of $(X_0, X_1)_{\theta,p}$ and from that they deduced results about exponential bases in Sobolev spaces. A particularly interesting observation is that in this case it is only the interpolation method and the K-functional of Γ that determines the result. That will also be true for the more general theorems presented in this article. In both [13,4], two indices (the definitions of which can be found in Section 5) were calculated from $K(t, \Gamma, \bar{X}')$ and the result for the (θ, p)-method is determined by comparing θ with the indices. Call the indices δ_0 and σ_0. $0 \leqslant \sigma_0 \leqslant \delta_0 \leqslant 1$ and the result is that

1. If $\theta < \sigma_0$ it holds that

$$(X_0, X_1 \cap \ker\Gamma)_{\theta,p} \approx (X_0, X_1)_{\theta,p}.$$

2. If $\theta > \delta_0$ it holds that Γ is bounded on $(X_0, X_1)_{\theta,p}$ and

$$(X_0, X_1 \cap \ker\Gamma)_{\theta,p} \approx (X_0, X_1)_{\theta,p} \cap \ker\Gamma.$$

3. If $\sigma_0 \leqslant \theta \leqslant \delta_0$ it follows that $(X_0, X_1 \cap \ker\Gamma)_{\theta,p}$ is not a closed subspace of $(X_0, X_1)_{\theta,p}$.

Ivanov and Kalton [4] proved this result and Löfström [11] proved the same result except that he did not give the answer for $\theta \in \{\sigma_0, \delta_0\}$. The proofs are different and were produced independently.

The article by Ivanov and Kalton [4], will be the foundation for the theory constructed in this article and we will show that their arguments can be used to prove more general results. In [4] they described the interpolation spaces they studied with the J-method, that is

$$(X_0, X_1 \cap \ker \Gamma)_{\theta,p} = \{x \in \Sigma(\bar{X}) \,|x = \sum_{k=-\infty}^{\infty} x_k,\ x_k \in \ker \Gamma \cap \Delta(\bar{X}),$$

$$\|\{\max(\|x_k\|_{X_0}, 2^k \|x_k\|_{X_1}) 2^{-\theta k})\}\|_{\ell^p} < \infty\}, \tag{1}$$

where the sum converges in $\Sigma(X_0, X_1 \cap \ker \Gamma)$. That is equivalent to demanding that the sum should converge in $\Sigma(\bar{X})$, since $\Sigma(X_0 \cap \ker \Gamma, X_1) \approx \Sigma(\bar{X})$ which is proved in Lemma 4.1. Now, we can make the observation that the assumption that Γ is bounded on at least one of the endpoint spaces is not necessary for the space in (1) (with $\Sigma(\bar{X})$-convergence) to be well defined. We only need to assume that $\Gamma \in (\Delta(\bar{X}))'$. We will denote the space in (1) by $J_{\theta,p;\Gamma}(\bar{X})$ and we will also use the notation $J_{\theta,p}(\bar{X})$ for $(X_0, X_1)_{\theta,p}$. Since it clearly holds that

$$J_{\theta,p;\Gamma}(\bar{X}) \subset J_{\theta,p}(\bar{X})$$

we can also in this more general situation ask the question about when $J_{\theta,p;\Gamma}(\bar{X})$ is a closed subspace of $J_{\theta,p}(\bar{X})$. In this situation, we will see that we get four indices in the interval $[0, 1]$ instead of two. Let us call them $\sigma_0, \delta_0, \delta_1, \sigma_1$ where σ_0 is always the smallest and σ_1 is always the largest. Their definitions can be found in Section 5. Under the extra assumption that $\delta_0 \leqslant \delta_1$ we can give a complete answer for regular couples and that is as follows:

1. If $\theta < \sigma_0$ or $\theta > \sigma_1$ it holds that
$$J_{\theta,p;\Gamma}(\bar{X}) \approx J_{\theta,p}(\bar{X}).$$
2. If $\delta_0 < \theta < \delta_1$ it holds that Γ is bounded on $J_{\theta,p}(\bar{X})$ and
$$J_{\theta,p;\Gamma}(\bar{X}) \approx J_{\theta,p}(\bar{X}) \cap \ker \Gamma.$$
3. If $\sigma_0 \leqslant \theta \leqslant \delta_0$ or $\delta_1 \leqslant \theta \leqslant \sigma_1$ it follows that $J_{\theta,p;\Gamma}(\bar{X})$ is not a closed subspace of $J_{\theta,p}(\bar{X})$.

So, what are these new spaces good for? The point with them is that if we have bounded linear maps $T_0 : X_0 \to Y_0$ and $T_1 : X_1 \to Y_1$ that agree on $\Delta(\bar{X}) \cap \ker \Gamma$ but not on $\Delta(\bar{X})$, as maps from $\Delta(\bar{X})$ to $\Sigma(\bar{Y})$, then we can not interpolate with the $J_{\theta,p}$-method to get a map $T : J_{\theta,p}(\bar{X}) \to J_{\theta,p}(\bar{Y})$ but we will see that if $\theta > \delta_0$ or $\theta < \delta_1$ we get an interpolated bounded map $T : J_{\theta,p;\Gamma}(\bar{X}) \to J_{\theta,p}(\bar{Y})$. This is especially interesting when $J_{\theta,p;\Gamma}(\bar{X})$ is a closed subspace of $J_{\theta,p}(\bar{X})$ with equivalent norms. We will also define spaces $J_{E;\Gamma}(\bar{X})$ for the general real method and not only for the (θ, p)-method and we will also in that case find assumptions that allow us to interpolate operators that only agree on $\Delta(\bar{X}) \cap \ker \Gamma$. In the author's Ph.D. Thesis [16], the results were also generalized to finitely many functionals. In [7,8], this kind of interpolation was applied to the study of Hardy-type inequalities initiated in [9] and to interpolation of Banach algebras. The theory developed in this paper will in the last section allow us to answer a question connected to some results from [9].

2. Preliminaries

A Banach couple $\bar{X}$ consists of two Banach spaces X_0 and X_1 continuously embedded into a Hausdorff topological vector space Ω. Given a Banach couple $\bar{X}$ we define two more spaces, $\Delta(\bar{X}) = X_0 \cap X_1$ and $\Sigma(\bar{X}) = X_0 + X_1$. $\Delta(\bar{X})$ and $\Sigma(\bar{X})$ are equipped with the norms

$$\|x\|_{\Delta(\bar{X})} = \max(\|x\|_{X_0}, \|x\|_{X_1})$$

and

$$\|x\|_{\Sigma(\bar{X})} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid x_0 + x_1 = x\}.$$

For every $t > 0$ we can define other equivalent norms on these spaces by renorming X_1. These norms are

$$\|x\|_{\Delta_t(\bar{X})} = J(t, x, \bar{X}) = \max(\|x\|_{X_0}, t\|x\|_{X_1})$$

and

$$\|x\|_{\Sigma_t(\bar{X})} = K(t, x, \bar{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} \mid x_0 + x_1 = x\}.$$

The functions $J(\cdot, x, \bar{X})$ and $K(\cdot, x, \bar{X})$ above are usually called the *J*- and *K*-functionals.

We will say that $\bar{X}$ is regular if $\Delta(\bar{X})$ is dense in both X_0 and X_1. If $\bar{X}$ is regular it follows that X_0' and X_1' are naturally embedded into $\Delta(\bar{X})'$ and by choosing those embeddings we define the dual couple $\bar{X}'$. It holds that $\Sigma(\bar{X}') = (\Delta(\bar{X}))'$ and $\Delta(\bar{X}') = (\Sigma(\bar{X}))'$. Similar identities hold by definition for the Δ_t and Σ_t spaces.

If $\bar{X}$ and $\bar{Y}$ are Banach couples we say that a pair of linear and bounded maps, $T_0 : X_0 \to Y_0$ and $T_1 : X_1 \to Y_1$, constitutes a couple map $T : \bar{X} \to \bar{Y}$ if they, as maps into $\Sigma(\bar{Y})$, agree on the intersection. The vector space $L(\bar{X}, \bar{Y}) = \{T : \bar{X} \to \bar{Y}\}$ with the norm $\|T\| = \max(\|T_0\|, \|T_1\|)$ is a Banach space. We will also use the notation $L(\bar{X}) = L(\bar{X}, \bar{X})$.

If $\bar{X}$ is a Banach couple and X is a Banach space with the property that $\Delta(\bar{X}) \subset X \subset \Sigma(\bar{X})$, where $\subset$ means continuous inclusion, then we say that X is an intermediate space for $\bar{X}$. If it also holds that $T : X \to X$ is bounded for all $T \in L(\bar{X})$ we say that X is an interpolation space.
The *K*- and *J*-functionals can be used to construct interpolation spaces with the so called *K*- and *J*- method. They are equivalent to each other and are often just referred to as the real method.

First we will define what equivalent means. We say that two functions f and g are equivalent if there are positive constants c_1 and c_2 such that

$$c_1 f \leqslant g \leqslant c_2 f.$$

Two Banach spaces are equivalent if we can identify them as vector spaces and the norms are equivalent. We will denote equivalence by $\approx$.

In this paper we will choose to work with the discrete versions of the *J*- and *K*-method. First let $\bar{\ell}_p = (\ell_p, \ell_p(2^{-k}))$ where ℓ_p is defined on $\mathbb{Z}$ and

$$\|\{\alpha_k\}\|_{\ell_p(2^{-k})} = \begin{cases} (\sum |\alpha_k 2^{-k}|^p)^{1/p} & p < \infty, \\ \sup |\alpha_k 2^{-k}| & p = \infty. \end{cases}$$

Let E be an interpolation space for $\bar{\ell_1}$. Then let

$$J_E(\bar{X}) = \left\{ x \in \Sigma(\bar{X}) \,\middle|\, x = \sum_{k=-\infty}^{\infty} x_k,\ x_k \in \Delta(\bar{X}),\ \|\{J(2^k, x_k)\}\|_E < \infty \right\}$$

and the norm is the infimum over all such representations. If $\Delta(\bar{\ell_1})$ is dense in E it follows that $\Delta(\bar{X})$ is dense in $J_E(\bar{X})$ for all Banach couples $\bar{X}$ and we say that E is a regular parameter for the discrete J-method. If D is an interpolation space to $\bar{\ell_\infty}$, K_D is defined by

$$\|x\|_{K_D(\bar{X})} = \|\{K(2^k, x)\}\|_D,\ x \in \Sigma(\bar{X}).$$

A parameter for the J-method is called non-degenerate if it is not contained in $\ell_1 \cup \ell_1(2^{-k})$ and a parameter for the K-method is called non-degenerate if it is not contained in $\ell_\infty \cup \ell_\infty(2^{-k})$. If D is a non-degenerate parameter for the discrete K-method and E is a non-degenerate parameter for the discrete J-method it holds that $K_D(\bar{X}) \approx J_{K_D(\bar{\ell_1})}(\bar{X})$ and $J_E(\bar{X}) \approx K_{J_E(\bar{\ell_\infty})}(\bar{X})$ for all Banach couples $\bar{X}$.

3. An algebraic construction

In this section, we present an algebraic construction that we will need in some proofs in the next section.

Definition 3.1. If $\bar{X}$ is a Banach couple and $\Gamma \in (\Delta\bar{X})'$ we define the Banach couple ${}_\Gamma\bar{X}$ by letting it consist of X_0 and X_1 embedded into the space $(X_0 \oplus X_1)/M$ where

$$M = \{(x_0, x_1) \mid \{x_0, x_1\} \subset \Delta(\bar{X}) \cap \ker\, \Gamma, x_0 + x_1 = 0 \text{ in } \Sigma(\bar{X})\}$$

which will then coincide with $\Sigma({}_\Gamma\bar{X})$.

Remark 3.1. Note that $\Sigma(\bar{X}) = (X_0 \oplus X_1)/\tilde{M}$ where

$$\tilde{M} = \{(x_0, x_1) \mid \{x_0, x_1\} \subset \Delta(\bar{X})\ x_0 + x_1 = 0 \text{ in } \Sigma(\bar{X})\}$$

is a quotient space of $\Sigma({}_\Gamma\bar{X})$ since $M \subset \tilde{M}$.

Remark 3.2. If $\bar{X}$ is a Banach couple and $\Gamma \in (\Delta(\bar{X}))'$ it holds that

$$\begin{aligned}\Delta({}_\Gamma\bar{X}) &= \{(x_0, x_1) \in X_0 \oplus X_1 \mid x_0 - x_1 = 0 \text{ in } \Sigma({}_\Gamma\bar{X})\}\\ &= \{(x_0, x_1) \in X_0 \oplus X_1 \mid x_0 - x_1 = 0 \text{ in } \Sigma(\bar{X}),\ \{x_0, x_1\} \subset \Delta(\bar{X}) \cap \ker\, \Gamma\}\\ &= \Delta(\bar{X}) \cap \ker\, \Gamma.\end{aligned}$$

In $L({}_\Gamma\bar{X}, \bar{X})$ there is an especially important map, namely the map consisting of the identity maps on X_0 and X_1. We will denote that map with Q because the induced map $Q : \Sigma({}_\Gamma\bar{X}) \to \Sigma(\bar{X})$ is a quotient map. Let s_0 and s_1 be the embeddings of X_0 and X_1 into $\Sigma({}_\Gamma\bar{X})$. $Q(s_0(x_0) + s_1(x_1)) = 0$ if and only if $x_0 + x_1 = 0$ in $\Sigma(\bar{X})$ since this is the image of $s_0(x_0) + s_1(x_1)$. Furthermore, $s_0(x) + s_1(-x) = 0$ for a certain $x \in \Delta(\bar{X})$ if and only if

$x \in \Delta({}_\Gamma\bar{X}) = \Delta(\bar{X}) \cap \ker \Gamma$. It follows that $s_0(x) + s_1(-x) = s_0(y) + s_1(-y)$ if and only if $\Gamma(x) = \Gamma(y)$ and therefore the kernel of Q is one dimensional and it is spanned by the element $u = s_0(x) - s_1(x)$ where x is any element in $\Delta(\bar{X})$ with $\Gamma(x) = 1$ and as in [16] we will refer to u as the predual of Γ.

Theorem 3.1. *Let $\bar{X}$ be a regular Banach couple, let $\Gamma \in (\Delta(\bar{X}))'$ and let u be the predual of Γ. It then follows that*

$$1 \leqslant K(t, u, {}_\Gamma\bar{X})K(1/t, \Gamma, \bar{X}') \leqslant 2.$$

Proof. Since all decompositions of u are of the form $u = s_0(x) + s_1(-x)$ where $x \in \Delta(\bar{X})$ and $\Gamma(x) = 1$ it follows that

$$\begin{aligned} K(t, u, {}_\Gamma\bar{X}) &= \inf\{\|x\|_0 + t\|x\|_1 \mid x \in \Delta(\bar{X}),\ \Gamma(x) = 1\} \\ &= \inf\left\{\frac{1}{|\Gamma(x)|};\ \|x\|_0 + t\|x\|_1 \leqslant 1\right\}. \end{aligned}$$

Since $K(t, \Gamma, \bar{X}') = \sup\{|\Gamma(x)| \mid J(t, x, \bar{X}) \leqslant 1\}$ and $J(t, x, \bar{X}) \leqslant \|x\|_0 + t\|x\|_1 \leqslant 2 J(t, x, \bar{X})$ the result now follows. $\square$

4. The $J_{E;\Gamma}$-functor

Let E be a regular parameter for the discrete J-method, $\bar{X} = (X_0, X_1)$ a regular Banach couple, Γ a bounded linear functional on $\Delta(\bar{X})$ and $\bar{\ell_1} = (\ell_1, \ell_1(2^{-k}))$. Define $J_{E;\Gamma}(\bar{X}) \subset J_E(\bar{X})$ by

$$J_{E;\Gamma}(\bar{X}) = \{x \in \Sigma\bar{X} \mid x = \sum_{k=-\infty}^{\infty} x_k,\ x_k \in \ker \Gamma,\ \{J(2^k, x_k)\} \in J_E(\bar{\ell_1})\}$$

and let the norm be the infimum of $\|\{J(2^k, x_k)\}\|$ over all such representations. For the (θ, p)-method we will write $J_{\theta,p}$ and $J_{\theta,p;\Gamma}$. We will begin our investigation of $J_{E;\Gamma}$ by stating some basic results. They follow easily from the definition and full proofs can be found in [16] on pp. 49–50.

Property 1. *If $\bar{X}$ is a Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and E is a parameter for the discrete J-method it follows that $J_{E;\Gamma}(\bar{X})$ is a Banach space and if E is a regular parameter it holds that $\Delta(\bar{X}) \cap \ker \Gamma$ is dense in $J_{E;\Gamma}(\bar{X})$.*

Property 2. *Let $\bar{X}$ be a Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let E be a regular parameter for the discrete J-method. Then the closure of $J_{E,\Gamma}(\bar{X})$ in $J_E(\bar{X})$ is $J_E(\bar{X}) \cap \ker \Gamma$ if Γ is bounded in the $J_E(\bar{X})$-norm and the whole of $J_E(\bar{X})$ if Γ is unbounded.*

The following theorem contains the result that makes the spaces $J_{E;\Gamma}(\bar{X})$ interesting.

Theorem 4.1. *Suppose that $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ are regular Banach couples, that $\Gamma \in (\Delta(\bar{X}))'$, that E is a non-degenerate parameter for the discrete J-method and that*

$$\left\{ \frac{1}{K(2^{-n}, \Gamma, \bar{X}')} \right\}_n \notin J_E(\bar{\ell_\infty}). \tag{2}$$

If $T_0 : X_0 \to Y_0$ and $T_1 : X_1 \to Y_1$ are bounded linear operators which, when considered as maps from $\Delta(\bar{X})$ to $\Sigma(\bar{Y})$, agree on $\Delta(\bar{X}) \cap \ker \Gamma$, then there is a bounded linear map $T : J_{E;\Gamma}(\bar{X}) \to J_E(\bar{Y})$ such that T as a map from $\Delta(\bar{X}) \cap \ker \Gamma$ to $\Sigma(\bar{Y})$ agrees with T_0 and T_1.

Proof. (T_0, T_1) constitutes a couple map $T : {}_\Gamma\bar{X} \to \bar{Y}$ and therefore interpolates to a map $T : J_E({}_\Gamma\bar{X}) \to J_E(\bar{Y})$. By Theorem 3.1 it holds that

$$1 \leqslant K(t, u, {}_\Gamma\bar{X})K(1/t, \Gamma, \bar{X}') \leqslant 2,$$

where u is the predual of Γ. Hence, it follows from (2) that $u \notin J_E({}_\Gamma\bar{X}) = K_{J_E(\ell_\infty)}({}_\Gamma\bar{X})$ and therefore $Q : J_E({}_\Gamma\bar{X}) \to J_{E;\Gamma}(\bar{X})$ is an isomorphism and we get our desired map by composing T with Q^{-1}. □

Lemma 4.1. *If $\bar{X}$ is a Banach couple where $\Delta(\bar{X}) \neq \{0\}$ and $\Gamma \in X_0'$ it holds that $\Sigma(X_0 \cap \ker \Gamma, X_1) \approx \Sigma(\bar{X})$ and if Γ is also in X_1' it holds that $\Gamma \in (\Sigma(\bar{X}))'$ and $\Sigma(X_0 \cap \ker \Gamma, X_1 \cap \ker \Gamma) \approx \Sigma(\bar{X}) \cap \ker \Gamma$.*

Proof. It is obviously true that $\|\cdot\|_{\Sigma(\bar{X})} \leqslant \|\cdot\|_{\Sigma(X_0 \cap \ker \Gamma, X_1)}$ so we only need to prove that there is a constant C such that $\|\cdot\|_{\Sigma(X_0 \cap \ker \Gamma, X_1)} \leqslant C\|\cdot\|_{\Sigma(\bar{X})}$.

Assume that Γ is bounded on X_0, that $x = x_0 + x_1$ and that

$$\|x\|_{\Sigma(\bar{X})}(1+\varepsilon) \geqslant \|x_0\|_0 + \|x_1\|_1.$$

Take $w \in \Delta(\bar{X})$ with $\Gamma(w) = 1$ and $J(1, w) \leqslant \frac{2}{K(1,\Gamma)}$. Then it follows that

$$\begin{aligned} \|x\|_{\Sigma(X_0 \cap \ker \Gamma, X_1)} &\leqslant \|x_0 - \Gamma(x_0)w\|_0 + \|x_1 + \Gamma(x_0)w\|_1 \\ &\leqslant \|x_0\|_0 + \|x_1\|_1 + |\Gamma(x_0)|(\|w\|_0 + \|w\|_1) \\ &\leqslant \|x_0\|_0 + \|x_1\|_1 + \frac{4}{K(1,\Gamma)}\|\Gamma\|_0\|x_0\|_0 \leqslant C\|x\|_{\Sigma(\bar{X})}. \end{aligned}$$

Thus $J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker \Gamma, X_1)$ and in the same way $J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker \Gamma, X_1 \cap \ker \Gamma)$ when Γ is also bounded on X_1. □

Proposition 4.1. *If $\bar{X}$ is a Banach couple with $\Delta(\bar{X}) \neq \{0\}$, $\Gamma \in X_0'$ and E is a regular parameter for the discrete J-method it holds that*

$$J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker \Gamma, X_1)$$

and if Γ is also bounded on X_1 it holds that

$$J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker \Gamma, X_1 \cap \ker \Gamma) \approx J_E(\bar{X}) \cap \ker \Gamma.$$

Proof. To prove that $J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker\, \Gamma, X_1)$ when $\Gamma \in X_0'$ we need to prove that $\sum_{k=-\infty}^{\infty} x_k$ converges in $\Sigma(\bar{X})$ iff $\sum_{k=-\infty}^{\infty} x_k$ converges in $\Sigma(X_0 \cap \ker\, \Gamma, X_1)$. That holds because $\Sigma(X_0 \cap \ker\, \Gamma, X_1) \approx \Sigma(\bar{X})$ and $J_{E;\Gamma}(\bar{X}) = J_E(X_0 \cap \ker\, \Gamma, X_1 \cap \ker\, \Gamma)$ when Γ is also bounded on X_1 because then Γ is bounded on $\Sigma(\bar{X})$ and $\Sigma(X_0 \cap \ker\, \Gamma, X_1 \cap \ker\, \Gamma) \approx \Sigma(\bar{X}) \cap \ker\, \Gamma$. That $J_E(X_0 \cap \ker\, \Gamma, X_1 \cap \ker\, \Gamma) \approx J_E(\bar{X}) \cap \ker\, \Gamma$ when $\Gamma \in (\Sigma\bar{X})'$ is a well-known fact. $\square$

We will continue by finding conditions for when $J_{E;\Gamma}(\bar{X})$ is closed in $J_E(\bar{X})$. Define the space $G = G_{E,K(\cdot,\Gamma)}$ by letting it consist of all sequences $\{\alpha_k\}$ with

$$\|\{\alpha_k\}\|_G = \| \left\{ \frac{\alpha_k}{K(2^{-k}, \Gamma)} \right\} \|_E < \infty.$$

Since

$$\frac{| < \Gamma, u_k > |}{K(2^{-k}, \Gamma)} \leqslant J(2^k, u_k)$$

it follows that:

$$\{J(2^k, u_k)\}_k \in E \Rightarrow \{< \Gamma, u_k >\}_k \in G$$

and that is a reason for studying G. Another reason is stated in the following lemma.

Lemma 4.2. *If $\bar{X}$ is a regular Banach couple, E is a regular parameter for the discrete J-method and $\Gamma \in (\Delta(\bar{X}))'$ it holds that*

$$\Gamma \in J_E(\bar{X})' \iff \phi \in G',$$

where ϕ is defined by the formula

$$\phi(\{\alpha_k\}) = \sum_{k=-\infty}^{\infty} \alpha_k.$$

Proof. The result follows from the fact [3, Theorem 3.7.2] that

$$\begin{aligned}\|\Gamma\|_{J_E(\bar{X})'} &= \sup\left\{ \left| \sum_{k=-\infty}^{\infty} K(2^k, \Gamma)\beta_{-k} \right| ;\ \|\{\beta_k\}\|_E \leqslant 1 \right\} \\ &= \sup\left\{ \left| \sum_{k=-\infty}^{\infty} \gamma_{-k} \right| ;\ \left\| \left\{ \frac{\gamma_k}{K(2^{-k}, \Gamma)} \right\} \right\|_E \leqslant 1 \right\} = \|\phi\|_{G'}. \qquad \square\end{aligned}$$

Let (e_k) be the standard basis in G, let S be the shift operator defined by $S(e_k) = e_{k+1}\ \forall k$ and let $T = S - I$. The following two theorems are based on an idea from [4]. $R(T)$ will denote the range of $T = S - I$.

Theorem 4.2. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let E is a regular parameter for the discrete J-method. Let also G and T be defined as above. If*
(a) *$R(T) = G$ it follows that $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X})$ and if*

(b) *$R(T)$ is closed with codimension one in G it follows that $\Gamma \in (J_E(\bar{X}))'$ and $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X}) \cap \ker\, \Gamma$.*

Proof. The plan is that to every $x \in J_E(\bar{X})$ in (a) and $x \in J_E(\bar{X}) \cap \ker\, \Gamma$ in (b) and to every almost optimal representation $x = \sum_{k=-\infty}^{\infty} x_k$ find a representation $x = \sum_{k=-\infty}^{\infty} y_k$ such that $y_k \in \ker\, \Gamma$ and

$$\|\{J(2^k, y_k)\}\|_E \leqslant C\|\{J(2^k, x_k)\}\|_E,$$

where C is independent of x and the representations.

Since $R(T)$ is closed in G, there is a constant D such that for all $\beta \in R(T)$ there is $\alpha \in G$ such that $T\alpha = \beta$ and $\|\alpha\| \leqslant D\|T\alpha\|$. If $R(T)$ is closed with codimension one, then $R(T)$ is the kernel of ϕ which is defined by

$$\phi(\{\alpha_k\}) = \sum_{k=-\infty}^{\infty} \alpha_k$$

and it follows that Γ is bounded on $J_E(\bar{X})$ by Lemma 5.1. Now suppose that $x \in J_E(\bar{X})$ with norm 1 and in (b) that $\Gamma(x) = 0$. Then there exists (x_k) in $\Delta(\bar{X})$ such that $\sum_{k=-\infty}^{\infty} x_k = x$ and

$$\|J(2^k, x_k)\|_E \leqslant 2,$$

which implies that

$$\|(\Gamma(x_k))\|_G \leqslant 2.$$

In (b) we also have that

$$\sum_{k=-\infty}^{\infty} \Gamma(x_k) = 0.$$

Thus, we can find $\alpha = \{\alpha_k\} \in G$ such that $T(\alpha) = \{\Gamma(x_k)\}$ and $\|\alpha\|_G \leqslant 2D$. Then find elements $u_k \in \Delta(\bar{X})$ such that

$$J(2^k, u_k) \leqslant 2\,\frac{|\alpha_k|}{K(2^{-k}, \Gamma)}$$

and $\Gamma(u_k) = \alpha_k$. It follows that

$$\|\{J(2^k, u_k)\}\|_E \leqslant 2\|\alpha\|_G \leqslant 4D.$$

Define v_k by $v_k = u_{k-1} - u_k$. Then it follows that

$$\begin{aligned}\|\{J(2^k, v_k)\}\|_E &\leqslant \|\{J(2^k, u_{k-1}) + J(2^k, u_k)\}\|_E \\ &\leqslant 3\|\{J(2^k, u_k)\}\| \leqslant 12D\end{aligned}$$

and $\Gamma(v_k) = \alpha_{k-1} - \alpha_k = \Gamma(x_k)$ and $\sum_{k=-\infty}^{\infty} v_k = 0$. Thus, $x = \sum_{k=-\infty}^{\infty}(x_k - v_k)$ and $x \in J_{E;\Gamma}(\bar{X})$ with

$$\|x\|_{J_{E;\Gamma}(\bar{X})} \leqslant (12D + 2).$$

It follows that in case (a) we have $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X})$ and in case (b) $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X}) \cap \ker \Gamma$. $\square$

With an additional mild assumption we also get the converse.

Theorem 4.3. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let E is a regular parameter for the discrete J-method. Let also G and T be defined as before and assume that $\sum_{n=-\infty}^{\infty} \min(1, 2^n)\|S^n\|_{B(E)} < \infty$. Then if*

(a) *$J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X})$ it holds that $R(T) = G$ and if*

(b) *Γ is bounded on $J_E(\bar{X})$ and $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X}) \cap \ker \Gamma$ it holds that $R(T)$ is closed with codimension one in G.*

Proof. Assume that $J_{E;\Gamma}(\bar{X})$ is closed in $J_E(\bar{X})$. Then (a) $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X})$ if Γ is not bounded on $J_E(\bar{X})$ and (b) $J_{E;\Gamma}(\bar{X}) \approx J_E(\bar{X}) \cap \ker \Gamma$ if Γ is bounded. In both cases, there is a constant $D > 0$ such that $\|x\|_{J_{E;\Gamma}(\bar{X})} \leqslant D\|x\|_{J_E(\bar{X})}$. In case (b)

$$\phi(\{\alpha_k\}_k) = \sum_{k=-\infty}^{\infty} \alpha_k,$$

is continuous on G and $R(T) \subset \ker \phi$.

Assume that $a = (a_k) \in G$, $\|a\|_G = 1$ and in (b) that $\sum_{k=-\infty}^{\infty} a_k = 0$. Take $x_k \in \Delta(\bar{X})$ such that $\Gamma(x_k) = a_k$ and

$$J(2^k, x_k) \leqslant 2\,\frac{|a_k|}{K(2^{-k}, \Gamma)}.$$

Then define x by $x = \sum_{k=-\infty}^{\infty} x_k$. it follows that $\|x\|_{J_E(\bar{X})} \leqslant 2$ and in (b) that $\Gamma(x) = 0$. Now, we can find $y_n \in \Delta(\bar{X}) \cap \ker \Gamma$ such that $\sum_{k=-\infty}^{\infty} y_n = x$ and

$$\|\{J(2^k, y_k)\}\|_E \leqslant 4D.$$

Let $u_k = x_k - y_k$ and $v_n = \sum_{k=n+1}^{\infty} u_k = -\sum_{k=-\infty}^{n} u_k$. Then it follows that

$$\|\{J(2^k, u_k)\}\|_E \leqslant 4D + 2$$

and

$$\|\{\Gamma(v_k)\}\|_G \leqslant \|\{J(2^k, v_k)\}\|_E \leqslant C\|\{J(2^k, u_k)\}\|_E$$

where $C = \sum_{k=-\infty}^{\infty} \min(1, 2^k)\|S^k\|_{B(E)}$ since

$$\begin{aligned}
\|\{\|v_n\|_0\}\|_E &\leqslant \left\| \left\{ \sum_{k=-\infty}^{n} \|u_k\|_0 \right\}_n \right\|_E = \left\| \left\{ \sum_{k=0}^{\infty} \|u_{n-k}\|_0 \right\}_n \right\|_E \\
&= \left\| \sum_{k=0}^{\infty} \{\|u_{n-k}\|_0\}_n \right\|_E = \left\| \sum_{k=0}^{\infty} (S^k\{\|u_n\|_0\}_n) \right\|_E \\
&= \left\| \left(\sum_{k=0}^{\infty} S^k \right) \{\|u_n\|_0\}_n \right\|_E \leqslant \sum_{k=0}^{\infty} \|S^k\|_{B(E)} \|\{\|u_n\|_0\}\|_E
\end{aligned}$$

and

$$
\begin{aligned}
\|\{2^n\|v_n\|_1\}\|_E &\leqslant \left\| \left\{ 2^n \sum_{k=n+1}^{\infty} \|u_k\|_1 \right\}_n \right\|_E \\
&= \left\| \left\{ 2^n \sum_{k=-\infty}^{-1} \|u_{n-k}\|_1 \right\| \right\|_E \\
&= \left\| \sum_{k=-\infty}^{-1} \{2^n\|u_{n-k}\|_1\} \right\|_E = \left\| \sum_{k=-\infty}^{-1} S^k\{2^{n+k}\|u_n\|_1\} \right\|_E \\
&= \left\| \left(\sum_{k=-\infty}^{-1} 2^k S^k \right) \{2^n\|u_n\|_1\}_n \right\|_E \\
&\leqslant \sum_{k=-\infty}^{-1} 2^k \|S^k\|_{B(E)} \|\{2^n\|u_n\|_1\}\|_E .
\end{aligned}
$$

it now follows that $T(\{\Gamma(v_k)\}) = \{\Gamma(u_k)\} = \{\Gamma(x_k)\} = \{a_k\}$. $\square$

Because of the previous two propositions we know that we can study $R(T)$ and G instead of $J_{E;\Gamma}(\bar{X})$ and $J_E(\bar{X})$. We will now look at the special case when the parameter E is a weighted ℓ_1-space. Note that G depends on $\bar{X}$ only through $K(\cdot, \Gamma, \bar{X}')$.

Proposition 4.2. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let E be the parameter for the discrete J-method defined by $\|\{\alpha_k\}\|_E = \sum |\alpha_k|\zeta_k$ where $\zeta_{k+1} \leqslant \zeta_k \leqslant 2\zeta_{k+1}$. Then the space G is defined by $\|\{\beta_k\}\|_G = \sum |\beta_k| w_k$ where $w_k = \frac{\zeta_k}{K(2^{-k}, \Gamma)}$ and $T = S - I$ on G. Then*

- (a) *T is injective iff $\sum w_k = \infty$.*
- (b) *$R(T) = G$ if there is a constant C not depending on n such that $\sum_{k=n+1}^{\infty} w_k \leqslant C w_n$ $\forall n$ or $\sum_{k=-\infty}^{n-1} w_k \leqslant C w_n$ $\forall n$.*
- (c) *$R(T)$ is closed with codimension one if there is a constant C not depending on j such that $\sum_{k=0}^{j-1} w_k \leqslant C w_j$ and $\sum_{k=0}^{j-1} w_{-k} \leqslant C w_{-j}$ $\forall j$.*

Proof. (a)

$$
Ta = 0 \iff a = c \sum_{k=-\infty}^{\infty} e_k,
$$

so T is injective if and only if

$$
\sum_{k=-\infty}^{\infty} e_k \notin G.
$$

(b)

$$\sum_{k=n+1}^{\infty} w_k \leqslant C w_n \quad \forall n \Rightarrow \left\{ \sum_{j=-\infty}^{k} \alpha_j \right\} \in G \quad \forall \{\alpha_k\} \in G$$

since

$$\sum_{k=-\infty}^{\infty} \left| \sum_{j=-\infty}^{k} \alpha_j \right| w_k \leqslant \sum_{j=-\infty}^{\infty} \left(\sum_{k=j}^{\infty} w_k \right) |\alpha_j|.$$

It now follows that $R(T) = G$ because

$$T\left(\left\{ -\sum_{j=-\infty}^{k} \alpha_j \right\} \right) = \{\alpha_k\}.$$

In the same way

$$\sum_{k=-\infty}^{n-1} w_k \leqslant C w_n \Rightarrow \left\{ \sum_{j=k+1}^{\infty} \alpha_j \right\} \in G \quad \forall \{\alpha_k\} \in G$$

since

$$\sum_{k=-\infty}^{\infty} \left| \sum_{j=k+1}^{\infty} \alpha_j \right| w_k \leqslant \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j-1} w_k \right) |\alpha_j|$$

and it follows that $R(T) = G$ because

$$T\left(\left\{ \sum_{j=k+1}^{\infty} \alpha_j \right\} \right) = \{\alpha_k\}.$$

(c) The linear functional ϕ from Lemma 4.2 is bounded since

$$\left\| \{\alpha_k\} \right\|_G \geqslant \frac{w_0}{C} \left\| \{\alpha_k\} \right\|_{\ell_1}$$

and therefore $R(T) \subset \ker\ \phi$. Assume that $\{\alpha_k\} \in \ker\ \phi$ and let

$$a_k = \sum_{j=k+1}^{\infty} \alpha_j = -\sum_{j=-\infty}^{k} \alpha_j$$

it follows that

$$\begin{aligned}
\|(a_k)\|_G &= \sum_{k=-\infty}^{\infty}\left|\sum_{j=k+1}^{\infty}\alpha_j\right| w_k \\
&= \sum_{k=0}^{\infty}\left|\sum_{j=k+1}^{\infty}\alpha_j\right| w_k + \sum_{k=1}^{\infty}\left|\sum_{j=k}^{\infty}\alpha_{-j}\right| w_{-k} \\
&\leqslant \sum_{j=1}^{\infty}|\alpha_j|\left(\sum_{k=0}^{j-1}w_k\right) + \sum_{j=1}^{\infty}|\alpha_{-j}|\left(\sum_{k=1}^{j}w_{-k}\right) \leqslant (C+1)\|(\alpha_k)\|_G
\end{aligned}$$

and

$$T\{a_k\} = \{\alpha_k\}. \qquad \square$$

Remark 4.1. The statements in Proposition 4.2 above are also true when E is defined by

$$\|\{\alpha_k\}\|_E^p = \sum |\alpha_k|^p \zeta_k$$

where $1 \leqslant p < \infty$. The proof is the same just with Minkowski's inequality instead of the triangle inequality.

5. The classical real methods

In this section, we look at what our results in the previous section imply for the (θ, p)-method. In particular the result from [4] follows. In our more general case we get four indices that determines the answer compared to two indices in the old case where Γ is bounded on at least one of the endpoint spaces.

Definition 5.1. Let $\bar{X}$ be a Banach couple and $\Gamma \in (\Delta(\bar{X}))'$. Then define δ_0, δ_1, σ_0 and σ_1 by

$$\sigma_1 = \lim_{k\to\infty}\sup_{n}\frac{1}{k}\log_2\frac{K(2^{n+k},\Gamma)}{K(2^n,\Gamma)},\quad \delta_1 = \lim_{k\to-\infty}\inf_{n\leqslant 0}\frac{1}{k}\log_2\frac{K(2^{n+k},\Gamma)}{K(2^n,\Gamma)}$$
$$\delta_0 = \lim_{k\to\infty}\sup_{n\geqslant 0}\frac{1}{k}\log_2\frac{K(2^{n+k},\Gamma)}{K(2^n,\Gamma)},\quad \sigma_0 = \lim_{k\to-\infty}\inf_{n}\frac{1}{k}\log_2\frac{K(2^{n+k},\Gamma)}{K(2^n,\Gamma)}.$$

The following three propositions follow easily from the definition above. The full proofs can be found in [16, pp. 66–67].

Proposition 5.1. *Suppose that $\bar{X}$ is a regular Banach couple, $\Gamma_1 \in (\Delta(\bar{X}))'$, $\Gamma_2 \in (\Delta(\bar{X}))'$ and that*

$$cK(t,\Gamma_1,\bar{X}') \leqslant K(t,\Gamma_2,\bar{X}') \leqslant CK(t,\Gamma_1,\bar{X}'),$$

where $0 < c \leqslant C$. It then holds that Γ_1 and Γ_2 have the same indices.

Proposition 5.2. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let σ_0, δ_0, δ_1 and σ_1 be defined as in Definition 5.1 above. Then it follows that*

$$\max\{\sigma_0, \delta_0, \delta_1, \sigma_1\} = \sigma_1$$

and

$$\min\{\sigma_0, \delta_0, \delta_1, \sigma_1\} = \sigma_0.$$

Proposition 5.3. *Let $\bar{X} = (X_0, X_1)$ be a regular Banach couple, let $\Gamma \in (\Delta(\bar{X}))'$ and let the indices $\sigma_0, \delta_0, \delta_1$ and σ_1 be defined as above. Also let $\bar{X}^r = (X_1, X_0)$ be the reversed couple of $\bar{X}$ and assume that $\tilde{\sigma}_0$, $\tilde{\delta}_0$, $\tilde{\delta}_1$ and $\tilde{\sigma}_1$ are the indices calculated from $K(t, \Gamma, (\bar{X}^r)')$. Then it holds that*

$$\tilde{\sigma}_1 = 1 - \sigma_0,\ \tilde{\sigma}_0 = 1 - \sigma_1$$

and

$$\tilde{\delta}_1 = 1 - \delta_0,\ \tilde{\delta}_0 = 1 - \delta_1.$$

In Theorem 4.1 we introduced an assumption that has to hold if we want to use a space $J_{E;\Gamma}(\bar{X})$ for interpolation purposes. In the proposition below we state that for the (θ, p)-method that holds if $\theta > \delta_0$ or $\theta < \delta_1$.

Proposition 5.4. *Assume that $\bar{X}$ is a regular Banach couple, that $\Gamma \in (\Delta(\bar{X}))'$, that $\theta \in (0, 1)$ and that $1 \leqslant p < \infty$. If $\theta > \delta_0$ or $\theta < \delta_1$ it follows that*

$$\left\| \left\{ \frac{1}{K(2^{-n}, \Gamma, \bar{X}')} \right\} \right\|_{\theta, p} = \infty.$$

Proposition 5.5. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let $\theta \in (0, 1)$ and $1 \leqslant p < \infty$.*
(a) *If $\theta > \sigma_1$ or $\theta < \sigma_0$ it follows that $J_{\theta,p;\Gamma}(\bar{X}) \approx J_{\theta,p}(\bar{X})$.*
(b) *If $\delta_0 < \theta < \delta_1$ it follows that Γ is bounded on $J_{\theta,p}(\bar{X})$ and $J_{\theta,p;\Gamma}(\bar{X}) \approx J_{\theta,p}(\bar{X}) \cap \ker \Gamma$.*

Proof. (a) The plan in this proof is to verify the assumptions from Proposition 4.2.

$$\theta > \sigma_1 \Rightarrow \exists \acute{\theta}\ :\ \theta > \acute{\theta} > \sigma_1.$$

It then holds that there is a constant K such that $k \geqslant K$ and $n \in Z$ implies that

$$\acute{\theta} > \frac{1}{k} \log_2 \frac{K(2^{n+k}, \Gamma)}{K(2^n, \Gamma)}$$
$$\Rightarrow \frac{2^{\acute{\theta}(k+n)}}{K(2^{n+k}, \Gamma)} > \frac{2^{\acute{\theta}(n)}}{K(2^n, \Gamma)} \Rightarrow w_{-(n+k)} 2^{(\acute{\theta}-\theta)k} > w_{-n},$$

which implies that there is a C such that $\sum_{k=n+1}^{\infty} w_k \leqslant C w_n \ \forall n$ where $w_n = \frac{2^{-\theta n}}{K(2^{-n},\Gamma)}$. In the same way $\theta < \sigma_0$ implies that there is a C such that

$$\sum_{k=-\infty}^{n-1} w_k \leqslant C w_n \quad \forall n.$$

(b) $\theta < \delta_1 \Rightarrow \exists \acute{\theta}$ such that $\theta < \acute{\theta} < \delta_1$ and $K < 0$ such that

$$\acute{\theta} < \frac{1}{k} \log_2 \frac{K(2^{n+k},\Gamma)}{K(2^n,\Gamma)} \quad \forall k \leqslant K \quad \forall n \leqslant 0 \Rightarrow w_{-(n+k)} > 2^{(\theta-\acute{\theta})k} w_{-n},$$

which implies that there is a C such that $\sum_{k=0}^{j-1} w_k \leqslant C w_j$ and in the same way $\theta > \delta_0 \Rightarrow \exists C$ s.t. $\sum_{k=0}^{j-1} w_{-k} \leqslant C w_j$. $\square$

Proposition 5.6. *Let $\bar{X}$ be a regular Banach couple, $\Gamma \in (\Delta(\bar{X}))'$ and let $\theta \in (0,1)$ and $1 \leqslant p < \infty$. If $\max(\delta_0,\delta_1) < \theta < \sigma_1$ or $\sigma_0 < \theta < \min(\delta_0,\delta_1)$ then $J_{\theta,p}(\bar{X})$ is not closed in $J_{\theta,p}(\bar{X})$.*

Proof. The proof is based on a method from [4].

Case (i) $\max(\delta_0,\delta_1) < \theta < \sigma_1$.

$\theta > \delta_0 \Rightarrow \sum_{k=0}^{\infty} w_{-k} = \infty \Rightarrow T$ is injective. Assume that $R(T)$ is closed in G. The plan is to prove that this implies that $\theta \leqslant \delta_1$. Since $R(T)$ is closed and T is injective there is a constant $C > 0$ such that $\|Ta\| \geqslant C\|a\| \ \forall a \in G$.

$\theta < \sigma_1 \Rightarrow \exists k > 9C^{-2}$ and $n \in Z$ such that $w_{n+k} > w_n$. Let

$$\alpha = (I + S + \ldots S^k)^2 e_n.$$

Since

$$(I + S + \ldots S^k)^2 = \sum_{j=0}^{2k} \gamma_j S^j,$$

where $\gamma_k = k+1$ for the fixed number k from above, it follows that $\|\alpha\| \geqslant k w_{n+k}$. We also get that

$$\begin{aligned} T^2\alpha &= (S-I)^2(I+S+\ldots S^k)^2 e_n = ((S-I)(I+S+\ldots+S^k))^2 e_n \\ &= (S^{k+1}-I)^2 e_n = e_n - 2e_{n+k+1} + e_{n+2k+2}. \end{aligned}$$

It now follows that

$$\begin{aligned} C^2 k w_{n+k} &\leqslant \|T^2\alpha\| = w_n + 2w_{n+k+1} + w_{n+2k+2} \\ &\leqslant w_n + 4w_{n+k} + 4w_{n+2k} \leqslant 9\max(w_n, w_{n+k}, w_{n+2k}) \\ &= 9\max(w_{n+k}, w_{n+2k}), \end{aligned}$$

which implies that $w_{n+k} < w_{n+2k}$ since $c^2k > 9$. By iteration it follows that $(w_{n+rk})_{r=0}^{\infty}$ is monotone increasing. Now for all large N and $j \geqslant 0$ it follows that

$$\begin{aligned}\frac{K(2^{-j},\Gamma)}{K(2^{-(j+N)},\Gamma)} &\geqslant \frac{K(2^{-(n+r_1k)},\Gamma)}{K(2^{-(n+r_2k)},\Gamma)}\\ &= \frac{2^{-(k+r_2k)\theta}}{K(2^{-(n+r_2k)},\Gamma)}\,\frac{K(2^{-(n+r_1k)},\Gamma)}{2^{-(k+r_1k)\theta}}\,2^{(r_2-r_1)k\theta}\\ &= w_{k+r_2k}\,\frac{1}{w_{k+r_1k}}\,2^{(r_2-r_1)k\theta} \geqslant 2^{(r_2-r_1)k\theta} \geqslant 2^{(N-2k)\theta},\end{aligned}$$

where $n+(r_1-1)k \leqslant j \leqslant n+r_1k$ and $n+r_2k \leqslant j+N \leqslant n+(r_2+1)k$. Thus

$$\inf_{j\geqslant 0}\frac{1}{N}\log_2\frac{K(2^{-j},\Gamma)}{K(2^{-(j+N)},\Gamma)} \geqslant \left(1-\frac{2k}{N}\right)\theta.$$

it now follows that $\theta \leqslant \delta_1$ by letting $N \to \infty$. Thus $R(T)$ is not closed.

Case (ii) follows from Case (i) and Proposition 5.3. □

The two propositions above does not solve the problem completely but with one more assumption we can get an answer for all $\theta \in (0,1)$.

Corollary 5.1. *If $\delta_0 \leqslant \delta_1$ it holds that $J_{\theta,p}(\bar{X})$ is not closed in$J_{\theta,p}(\bar{X})$ if and only if $\sigma_0 \leqslant \theta \leqslant \delta_0$ or $\delta_1 \leqslant \theta \leqslant \sigma_1$.*

Proof. It only remains to prove that $R(T)$ is not closed in G for the breakpoints. Define $\tau : \ell_1\left(\frac{1}{K(2^{-k},\Gamma)}\right) \to G$ by $\tau(\{\alpha_k\}) = \{2^{\theta k}\alpha_k\}$. Then τ is an isometric isomorphism and if $T_\theta = S - 2^{\theta} I$ it follows that $\tau(R(T_\theta)) = R(T)$. Now the result for the breakpoints follows from the fact that the set of Fredholm operators on $\ell_1\left(\frac{1}{K(2^{-k},\Gamma)}\right)$ is open. □

Now we will look at the case from [4] where $\Gamma \in X_0'$.

Corollary 5.2 (Ivanov and Kalton). *If $\theta \in (0,1)$, $1 \leqslant p < \infty$ and Γ is bounded on X_0 but not on X_1, then $J_E(X_0 \cap \ker\Gamma, X_1) \approx J_{\theta,p}(\bar{X})$ if $\theta > \sigma_1$, Γ is bounded on $J_{\theta,p}(\bar{X})$ and $J_E(X_0 \cap \ker\Gamma, X_1) \approx J_{\theta,p}(\bar{X}) \cap \ker\Gamma$ if $\theta < \delta_1$ and $J_E(X_0 \cap \ker\Gamma, X_1)$ is not closed in $J_{\theta,p}(\bar{X})$ if $\delta_1 \leqslant \theta \leqslant \sigma_1$.*

Proof. $K(t,\Gamma) \leqslant C \Rightarrow \delta_0 = \sigma_0 = 0$. □

6. An application to Hardy-type inequalities

Let

$$L_p(w) = \left\{ f \text{ on } (0,\infty) \mid \|f\|_{L_p(w)} = \left(\int_0^\infty |f(x)|^p w(x)\,dx\right)^{1/p} < \infty \right\},$$

$$C_p(w) = \left\{ f \text{ on } (0,\infty) \mid \|f\|_{C_p(w)} = \left(\int_0^\infty \left| \frac{1}{x} \int_0^x f(s)\,ds \right|^p w(x)\,dx \right)^{1/p} < \infty \right\}$$

and

$$N = \left\{ f \text{ on } (0,\infty) \,\middle|\, \int_0^\infty f(s)ds = 0 \right\}.$$

Krugljak et al. [9] investigated the interpolation of intersections question about when the formula

$$(N \cap L_{p_0}(w_0), N \cap L_{p_1}(w_1))_{\theta,p} \approx N \cap (L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p}$$

holds. They said that the question has positive answer if it holds and negative if it does not. Their reason for asking the question is related to the well-known result that if $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, then there is a constant $C(\alpha)$ such that

$$\int_0^\infty |u(s)| s^{\alpha-1}\,ds \leqslant C(\alpha) \int_0^\infty |u'(s)| s^\alpha\,ds \tag{3}$$

for all infinitely differentiable functions on $(0,\infty)$ with compact support. This result is implied by the Hardy inequalities

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(s)\,ds \right| x^\alpha\,dx \leqslant \frac{1}{|\alpha|} \int_0^\infty |f(x)| x^\alpha\,dx \ (\alpha < 0)$$

and

$$\int_0^\infty \left| \frac{1}{x} \int_x^\infty f(s)\,ds \right| x^\alpha dx \leqslant \frac{1}{|\alpha|} \int_0^\infty |f(x)| x^\alpha\,dx \ (\alpha > 0).$$

There is also a negative result that says that $C(\alpha) \to \infty$ as $\alpha \to 0$. Krugljak, Maligranda and Persson wanted to know why it is impossible to interpolate (3) between $\alpha = 1$ and $\alpha = -1$ and prove an inequality for $\alpha = 0$. Naturally, we cannot interpolate the inequalities directly since the inequalities contain two different operators H_+ and H_- defined by

$$(H_+ f)(x) = \frac{1}{x} \int_0^x f(s)\,ds$$

and

$$(H_- f)(x) = -\frac{1}{x} \int_x^\infty f(s)\,ds,$$

but they coincide on N so we can interpolate the inequalities between $N \cap L_1(x)$ and $N \cap L_1(x^{-1})$. In [9] they proved that

$$(N \cap L_1(x), N \cap L_1(x^{-1}))_{\theta,1} \approx N \cap (L_1(x), L_1(x^{-1}))_{\theta,p} \iff \theta \neq \tfrac{1}{2}$$

and therefore we do not get an inequality for $\alpha = 0$ by interpolating. For $\theta = \frac{1}{2}$ they found the answer that

$$(N \cap L_1(x), N \cap L_1(x^{-1}))_{1/2,1} \approx C_1(1) \cap L_1(1)$$

and they found a function $f \in (L_1 \backslash C_1) \cap N$. They also studied an example with the weights $w_0(x) = \max(x^{\alpha_0}, x^{\beta_0})$ and $w_1(x) = \min(x^{-\alpha_1}, x^{-\beta_1})$ where $0 < \alpha_0 \leqslant \alpha_1$, $0 < \beta_0 \leqslant \beta_1$ and $\alpha_0/\alpha_1 \leqslant \beta_0/\beta_1$. For $\theta \notin [\frac{\alpha_0}{\alpha_0+\beta_0}, \frac{\alpha_1}{\alpha_1+\beta_1}]$ they found that

$$(N \cap L_1(w_0), N \cap L_1(w_1))_{\theta,1} \approx N \cap L_1(w_0^{1-\theta} w_1^{\theta})$$

and for $\theta \in \left[\frac{\alpha_0}{\alpha_0+\beta_0}, \frac{\alpha_1}{\alpha_1+\beta_1}\right]$ they proved that

$$(N \cap L_1(w_0), N \cap L_1(w_1))_{\theta,1} \approx N \cap C_1(w_0^{1-\theta} w_1^{\theta}) \cap L_1(w_0^{1-\theta} w_1^{\theta}).$$

From that they concluded that

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(s)\, ds \right| w_0^{1-\theta} w_1^{\theta}\, dx \leqslant C \int_0^\infty |f(x)| w_0^{1-\theta} w_1^{\theta}\, dx \tag{4}$$

and

$$\int_0^\infty \left| \frac{1}{x} \int_x^\infty f(s)\, ds \right| w_0^{1-\theta} w_1^{\theta}\, dx \leqslant C \int_0^\infty |f(x)| w_0^{1-\theta} w_1^{\theta}\, dx \tag{5}$$

if $\theta \notin \left[\frac{\alpha_0}{\alpha_0+\beta_0}, \frac{\alpha_1}{\alpha_1+\beta_1}\right]$ and $f \in N$. Furthermore, it is known that (5) holds for all *f* when $\theta < \frac{\alpha_0}{\alpha_0+\beta_0}$ and (4) holds for all *f* when $\theta > \frac{\alpha_1}{\alpha_1+\beta_1}$. From the theory created in this paper we will deduce the new result that if $\theta \in \left[\frac{\alpha_0}{\alpha_0+\beta_0}, \frac{\alpha_1}{\alpha_1+\beta_1}\right]$ it holds that

$$N \cap C_1(w_0^{1-\theta} w_1^{\theta}) \cap L_1(w_0^{1-\theta} w_1^{\theta}) \approx N \cap L_1(w_0^{1-\theta} w_1^{\theta}).$$

For that purpose we prove the following lemma.

Lemma 6.1. *Assume that w_0 is an increasing continuous weight function and that w_1 is a decreasing continuous weight function such that $\frac{w_0}{w_1}$ is strictly increasing, $\lim_{x\to 0} \frac{w_0(x)}{w_1(x)} = 0$, $\lim_{x\to\infty} \frac{w_0(x)}{w_1(x)} = \infty$ and let $\bar{L_1} = (L_1(w_0), L_1(w_1))$. Define $\Gamma \in (\Delta(\bar{L_1}))'$ by letting*

$$< \Gamma, f >= \int_0^\infty f(s)\, ds$$

and let

$$s(t) = \left(\frac{w_0}{w_1}\right)^{-1}(t).$$

That is $s(t)$ is such that $w_0(s(t)) = t w_1(s(t))$. It then follows that

$$\frac{1}{w_0(s(1/t))} \leqslant K(t, \Gamma, \bar{L_1}') \leqslant \frac{2}{w_0(s(1/t))}.$$

Proof. Fix $t > 0$ and define the measure μ_t by letting

$$\mu_t = \frac{1}{w_0(s(1/t))}\, \delta_{s(1/t)}.$$

If we let $\bar{M} = (M(w_0), M(w_1))$ where $M(w)$ consists of all regular Borel measures on $(0, \infty)$ for which

$$\|\mu\|_{M(w)} = \int_0^\infty w(s)d|\mu|(s) < \infty.$$

It holds that

$$J(1/t, \mu_t, \bar{M}) = \frac{1}{w_0(s(1/t))} \max\left(w_0(s(1/t)), \frac{1}{t}w_1(s(1/t))\right) = 1.$$

If we choose non-negative functions $f_\varepsilon \in \Delta(\bar{L_1})$ such that $\int_0^\infty f_\varepsilon(s)\,ds = 1$ and $\operatorname{supp}(f_\varepsilon) \subset [s(1/t) - \varepsilon, s(1/t) + \varepsilon]$ it therefore holds that

$$\lim_{\varepsilon \to 0+} < \frac{f_\varepsilon}{J(1/t, f_\varepsilon, \bar{L_1})}, \Gamma >= \frac{1}{w_0(s(1/t))}.$$

Thus

$$K(t, \Gamma, \bar{L_1}') \geqslant \frac{1}{w_0(s(1/t))}.$$

To prove the upper estimate we will make a decomposition of Γ. Let

$$< \Gamma_0, f >= \int_{s(1/t)}^\infty f(s)\,ds$$

and

$$< \Gamma_1, f >= \int_0^{s(1/t)} f(s)\,ds.$$

It holds that $\Gamma = \Gamma_0 + \Gamma_1$ and

$$\|\Gamma_0\|_{L_1(w_0)'} = t\|\Gamma_1\|_{L_1(w_1)'} = \frac{1}{w_0(s(1/t))}.$$

The upper estimate follows from that. □

From the formula for $K(t, \Gamma, \bar{L_1}')$ in Lemma 6.1. we can calculate the four indices and use the results from Section 5 to determine the interpolation result. We will do this for the example that was studied in [9].

Theorem 6.1. *Let* $\bar{L_1} = (L_1(w_0), L_1(w_1))$ *where* $w_0(x) = \max(x^{\alpha_0}, x^{\alpha_1})$, $w_1(x) = \min(x^{-\beta_0}, x^{-\beta_1})$, $0 < \alpha_0 \leqslant \alpha_1$, $0 < \beta_0 \leqslant \beta_1$ *and* $\alpha_0/\alpha_1 \leqslant \beta_0/\beta_1$. *Define* $\Gamma \in (\Delta\bar{L_1})'$ *by letting* $\Gamma(f) = \int_0^\infty f(s)\,ds$. *If* $0 < \theta < 1$ *it follows that*

$$J_{\theta,1;\Gamma}(\bar{L_1}) \approx J_{\theta,1}(\bar{L_1}) \iff \theta \notin \left[\frac{\alpha_0}{\alpha_0 + \beta_0}, \frac{\alpha_1}{\alpha_1 + \beta_1}\right]$$

and

$$J_{\theta,1;\Gamma}(\bar{L_1}) \approx J_{\theta,1}(\bar{L_1}) \cap \ker \Gamma \text{ and } \Gamma \in (J_{\theta,1}(\bar{L_1}))'$$
$$\iff \theta \in \left(\frac{\alpha_0}{\alpha_0 + \beta_0}, \frac{\alpha_1}{\alpha_1 + \beta_1}\right).$$

Proof. First note that the assumptions implies that $\frac{\alpha_0}{\alpha_0+\beta_0} \leqslant \frac{\alpha_1}{\alpha_1+\beta_1}$. If we manage to prove that

$$K(t, \Gamma, (\bar{L_1})') \approx \min\left(t^{\frac{\alpha_0}{\alpha_0+\beta_0}}, t^{\frac{\alpha_1}{\alpha_1+\beta_1}}\right),$$

we would be finished since that implies the identities

$$\sigma_0 = \delta_0 = \frac{\alpha_0}{\alpha_0+\beta_0},\ \sigma_1 = \delta_1 = \frac{\alpha_1}{\alpha_1+\beta_1}.$$

The result then follows from Proposition 5.5 and Corollary 5.1.

If

$$s(t) = \min(t^{\frac{1}{\alpha_0+\beta_0}}, t^{\frac{1}{\alpha_1+\beta_1}})$$

it follows that

$$\max(s(t)^{\alpha_0}, s(t)^{\alpha_1}) = t\ \min(s(t)^{-\beta_0}, s(t)^{-\beta_1}).$$

Hence

$$K(t, \Gamma, \bar{L_1}') \approx \frac{1}{w_0(s(1/t))} = \min(t^{\frac{\alpha_0}{\alpha_0+\beta_0}}, t^{\frac{\alpha_1}{\alpha_1+\beta_1}})$$

and the statements of the theorem now follows as explained in the beginning of the proof. $\square$

Theorem 6.2. *Let* $w_0(x) = \max(x^{\alpha_0}, x^{\alpha_1})$, $w_1(x) = \min(x^{-\beta_0}, x^{-\beta_1})$, $0 < \alpha_0 \leqslant \alpha_1$, $0 < \beta_0 \leqslant \beta_1$, $\alpha_0/\alpha_1 \leqslant \beta_0/\beta_1$ *and that* $\frac{\alpha_0}{\alpha_0+\beta_0} < \theta < \frac{\alpha_1}{\alpha_1+\beta_1}$. *Then there is a constant C such that*

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(s)\, ds \right| w_0^{1-\theta} w_1^{\theta}\, dx \leqslant C \int_0^\infty |f(x)| w_0^{1-\theta} w_1^{\theta}\, dx \tag{6}$$

holds for all $f \in N$.

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